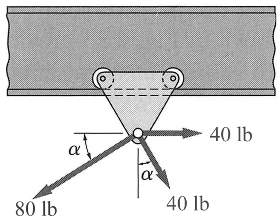
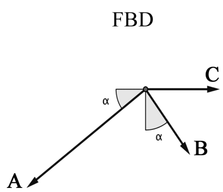
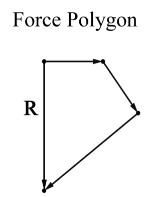
Lesson 3 Example 3

This is problem 2.32 in *Mechanics For Engineers: Statics*, 4th edition, by Beer and Johnston.

Find the value of α such that the resultant of the three forces is vertical.







$$\Sigma F_x = R_x = 0$$
$$-A_x + B_x + C = 0$$
$$-80\cos\alpha + 40\sin\alpha + 40 = 0$$
$$\sin\alpha - 2\cos\alpha + 1 = 0 \qquad (1)$$
$$\alpha = ???$$

To solve this problem you will need to solve equation (1) for α. It's not obvious how to do this using algebra so that's why I recommended finding the answer using trial and error or a graphing calculator. However, if you are curious to see how to solve it algebraically I have written up the solution here.

The issue is that it is difficult to untangle the angle buried in both the sin and cos terms in order to isolate α on one side of the equals sign. The way out is to use a trick known as the "Weierstrass t substitution", also called the "Miracle substitution" which is, according to one calculus textbook author, "The world's sneakiest substitution."

It works like this. Consider the unit circle intersected by a line passing through point A at $(-1, 0)$ and point D at (x, y) as shown in Fig. 1. It is easy to prove by basic geometry that $\angle OAC$ is half of $\angle BOD$, and by SOH-CAH-TOA that

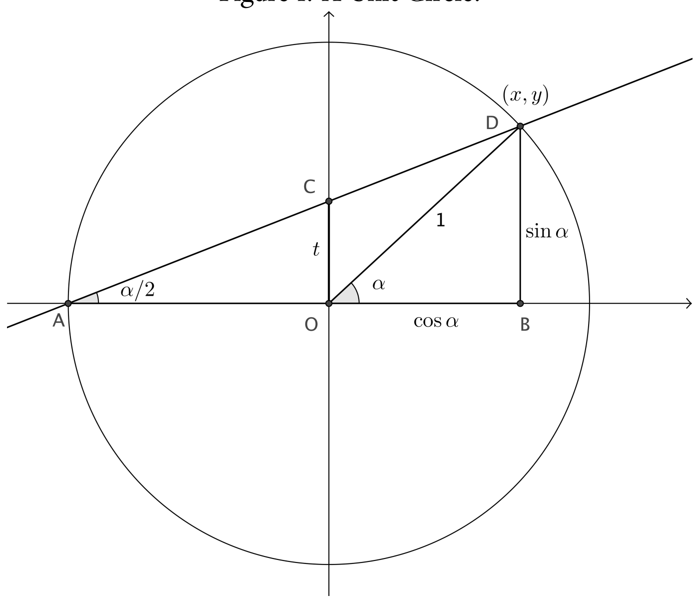
$$x = \cos\alpha \qquad (2)$$
$$y = \sin\alpha \qquad (3)$$
$$t = \tan(\alpha/2) \qquad (4)$$

The equation of line AD can be determined by noting that its slope is t and its y-intercept is also t, so its equation is

$$y = tx + t \qquad (5)$$
$$= t(x + 1) \qquad (6)$$

The equation for the unit circle is

Figure 1: A Unit Circle.



(7)
$$x^2 + y^2 = 1$$

Solving (6) and (7) simultaneously gives the coordinates of points A and D where the line intersects the circle.

$$\begin{aligned} x^2 + (tx + t)^2 &= 1 \\ x^2 + t^2x^2 + 2t^2x + t^2 &= 1 \\ (1 + t^2)x^2 + 2t^2x + (t^2 - 1) &= 0 \end{aligned}$$

Solve this by applying the quadratic equation

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-2t^2 \pm \sqrt{(2t^2)^2 - 4(1+t^2)(t^2-1)}}{2(t^2+1)} \\
&= \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^4-1)}}{2(t^2+1)} \\
&= \frac{-2t^2 \pm 2}{2(t^2+1)} \\
&= \frac{-t^2 \pm 1}{t^2+1}
\end{aligned}
$$

Which has two solutions

$$
x = \begin{cases} \dfrac{-t^2-1}{t^2+1} = \dfrac{-(t^2+1)}{(t^2+1)} = -1 \\[2ex] \dfrac{1-t^2}{1+t^2} \end{cases} \tag{8}
$$

The first corresponds to point A on the unit circle which is uninteresting to us; the second solution is the x coordinate of point D. With x expressed in terms of t, we can find the y coordinate of D with equation (6).

$$
\begin{aligned}
y &= t(x+1) \\
&= t\left(\frac{1-t^2}{1+t^2} + 1\right) \\
&= t\frac{(1-t^2)+(1+t^2)}{1+t^2} \\
&= \frac{2t}{1+t^2}
\end{aligned}
$$

Recalling (2), (3), and (4) the final Weierstrass substitution is:

$$\cos\alpha = \frac{1-t^2}{1+t^2} \qquad \text{and} \tag{9}$$

$$\sin\alpha = \frac{2t}{1+t^2} \qquad \text{where,} \tag{10}$$

$$t = \tan(\alpha/2) \tag{11}$$

Substituting (9) and (10) into (1) eliminates the trig functions of α and gives a rational function of t which is solvable.

$$
\begin{aligned}
\sin\alpha - 2\cos\alpha + 1 &= 0 \\
\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) + 1 &= 0 \\
\frac{2t - 2 + 2t^2 + 1 + t^2}{1+t^2} &= 0 \\
3t^2 + 2t - 1 &= 0 && \text{which factors to} \\
(3t-1)(t+1) &= 0 && \text{so,} \\
t &= \begin{cases} -1 \\ 1/3 \end{cases}
\end{aligned}
$$

Now that we know t, equation (11) can be used to find the answer, α.

$$
\begin{aligned}
\tan(\alpha/2) &= t & \tan(\alpha/2) &= t \\
\tan(\alpha/2) &= -1 & \tan(\alpha/2) &= 1/3 \\
\alpha/2 &= \tan^{-1}(-1) & \alpha/2 &= \tan^{-1}(1/3) \\
\alpha &= 2\tan^{-1}(-1) & \alpha &= 2\tan^{-1}(1/3) \\
\alpha &= 2(-45^\circ) & \alpha &\approx 2(18.43^\circ)
\end{aligned}
$$

$$
\alpha \begin{cases} = -90^\circ \\ \approx 36.87^\circ \end{cases} \tag{12}
$$

In addition to the angles found above, there are an infinite number of additional solutions found by adding integer multiples of 360° to these results, but these are the two closest to zero and $\alpha \approx 36.87^\circ$ is the value anticipated by the problem diagram.